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The $1/D$ Expansion for Classical Magnets: Low-Dimensional Models with Magnetic Field

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The field-dependent magnetization $m(H, T)$ of one- and two-dimensional classical magnets described by the D-component vector model is calculated analytically in the whole range of temperature and magnetic fields with the help of the $1/D$ expansion. In the first order in $1/D$ the theory reproduces with a good accuracy the temperature dependence of the zero-field susceptibility of antiferromagnets χ with maximum at $T \lesssim |J_0|/D$ (J_0 is the Fourier component of the exchange interaction) and describes for the first time the singular behavior of $\chi(H, T)$ at small temperatures and magnetic fields: $\lim_{T\to 0} \lim_{H\to 0} \chi(H, T) = 1/(2|J_0|)(1-1/D)$ and $\lim_{H\to 0} \lim_{T\to 0} \chi(H, T) = 1/(2|J_0|)$.

KEY WORDS: Low-dimensional magnets; magnetic susceptibility; spherical model; $1/D$ ($1/n$) expansion.

1. INTRODUCTION

A great variety of low-dimensional magnetic systems have been synthesized and experimentally investigated in recent decades (see, e.g., refs. 1 and 2). The idealized one- and two-dimensional models (without interplane or interchain coupling and anisotropy) are characterized by a strong short-range order in the low-temperature region, whereas the long-range order is ruled out as being smeared off by the long-wavelength spin waves. Complementary to the high-temperature series expansions (HTSE; see, e.g., refs. 3–5), such approaches as "modified spin-wave theory"[6] and "Schwinger boson mean-field theory"[7] were applied to low-dimensional antiferromagnets at low temperatures. These two methods giving very similar results (with a wrong factor in the antiferromagnetic susceptibility

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in ref. 7) are not rigorous expansions in the parameter $T/|J| \ll 1$ (J is the exchange integral), but rather are variational approaches. The results break down, however, at $T \sim |J|$ and thus cannot describe the situation in the whole temperature range. Since the absence of magnetization was introduced in ref. 6 as an additional self-consistency requirement, the generalization for the case with the external magnetic field **H** is a problem.

In ref. 8 an analytical method of calculation of the physical quantities of *classical* low-dimensional magnets in the whole temperature range was proposed, which is based on the $1/D$ expansion for the model of D-component classical spin vectors on a lattice[9]:

$$\mathscr{H} = -\mathbf{H} \sum_i \mathbf{m}_i - \tfrac{1}{2} \sum_{ij} J_{ij} \mathbf{m}_i \mathbf{m}_j, \qquad |\mathbf{m}| = 1 \tag{1.1}$$

with the help of the diagram technique developed in ref. 10. For the Heisenberg model ($D = 3$) in the first order in $1/D$ the calculated temperature dependences of the antiferromagnetic susceptibility and internal energy at $H = 0$ turn out to be very good, as shown by comparison with MC simulation data[11] for the internal energy of the square lattice (s.l.) classical ferromagnet and with the exact results[12] for a "toy" example of the classical linear chain (l.c.) model. In particular, for both models the characteristic maximum of the antiferromagnetic susceptibility at $T \lesssim |J_0|/D$, in contrast to refs. 6 and 7, is well reproduced. The reason for the efficiency of the $1/D$ expansion even for $D = 3$ is that it yields the *exact* results for the energy and antiferromagnetic susceptibility at $T \to 0$ and reproduces several leading terms of their HTSE expansions[8] interpolating thus between these limits in the whole temperature range. The applicability of the approach to classical low-dimensional magnets proposed in ref. 8 is not restricted to the case $H = 0$, and it can be applied to the problem of the singular behavior of the antiferromagnetic susceptibility $\chi^{\mathrm{AF}}(H, T)$ at low temperatures and magnetic fields, i.e., the inpermutability of its limits $\lim_{H \to 0} \lim_{T \to 0}$ and $\lim_{T \to 0} \lim_{H \to 0}$, which could not up to now be described by other analytical methods. The physical reason for such a singular behavior is the following. With lowering temperature the system becomes locally antiferromagnetically ordered, and $D - 1$ susceptibilities transverse with respect to the local orientation of the sublattice tend to the value $1/(2|J_0|)$ ($J_0 = zJ$, z is the number of nearest neighbors), whereas the longitudinal susceptibility tends to zero. At $H = 0$ there is no preferred direction, and the susceptibility of the sample is given by the average over the local orientations of sublattice magnetizations, which results in

$$\lim_{T \to 0} \lim_{H \to 0} \chi^{\mathrm{AF}}(H, T) = 1/(2|J_0|)(1 - 1/D)$$

For the Heisenberg model this D-dependent factor makes up the well-known number 2/3. To the contrary, for arbitrarily small $H \neq 0$ at sufficiently low T the lowest energy state with the sublattice magnetizations driven perpendicular to the field $\mathbf{H}$ and tilted in the direction of $\mathbf{H}$ is realized. In this state the susceptibility takes on its transverse value

$$\lim_{H \to 0} \lim_{T \to 0} \chi^{\mathrm{AF}}(H, T) = 1/(2\,|J_0|)$$

A quantitative description of this effect with the help of the $1/D$ expansion and the calculation of the magnetization $m(H, T)$ of low-dimensional classical antiferromagnets in the whole range of temperature and magnetic fields is the pupose of this work.

The remainder of this paper is organized as follows. In Section 2 an improved version of the diagrammatic $1/D$ expansion for classical spin systems with magnetic field is developed and the results for the magnetization and spin–spin correlation function in the first-order in $1/D$ are obtained. In Section 3 the general $1/D$ results, given by double integrals over the Brillouin zone, are calculated analytically and analyzed for the classical linear chain model, for which there is no exact solution in the case with magnetic field. In Section 4 the results are converted for two-dimensional systems into a form convenient for numerical calculations and analysis at low temperatures, and the temperature and magnetic field dependences of the antiferromagnetic susceptibility are represented. In Section 5 some important features of the $1/D$ expansion and its applicability to systems with $D = 2, 3$ are discussed.

2. THE 1/D EXPANSION

The physical quantities of ferro- and antiferromagnets described by the Hamiltonian (1.1) can be expanded in powers of $1/D$ with the help of the diagram technique for classical spin systems.[8,10] Here the consideration of ref. 8 is improved and generalized to the case $H \neq 0$. We choose the z axis along the magnetic field $\mathbf{H}$; the other (transverse) components of the spin vector $\mathbf{m}$ are designated by the index $\alpha = 2, 3, \ldots, D$. The wavevector-dependent transverse susceptibility $\chi_\perp(\mathbf{k}) \equiv \chi_\alpha(\mathbf{k})$ of a classical spin system is related to the Fourier transform of the spin–spin correlation function $S_{\alpha\alpha}(\mathbf{r} - \mathbf{r}') = \langle m_\alpha(\mathbf{r})\, m_\alpha(\mathbf{r}')\rangle$ via the formula $\chi_\perp(\mathbf{k}) = \beta S_{\alpha\alpha}(\mathbf{k})$, $\beta = 1/T$. With the help of the diagram technique of ref. 8, $\chi_\perp(\mathbf{k})$ can be represented as

$$\chi_\perp(\mathbf{k}) = \frac{\beta \hat{\Lambda}_{\alpha\alpha}(\mathbf{k})}{1 - \hat{\Lambda}_{\alpha\alpha}(\mathbf{k})\, \beta J_{\mathbf{k}}} \tag{2.1}$$

where $\hat{\Lambda}_{\alpha\alpha}(\mathbf{k})$ is the compact (irreducible) part of $S_{\alpha\alpha}(\mathbf{k})$ given by the diagrams, which cannot be cut by the one interaction line $\beta J_{\mathbf{k}}$. For isotropic systems considered here it is not necessary to write down the diagrams for the magnetization $m = \langle m_z \rangle$ because $m(H)$ can be determined from (2.1). Indeed, in a transverse magnetic field $H_\perp \ll H$ the magnetization $\mathbf{m}$ rotates simply by the angle $\theta = H_\perp / H \ll 1$, which results in the important relation

$$\chi_\perp \equiv \chi_\perp(0) = m/H \tag{2.2}$$

The longitudunal susceptibility can be determined now by the formula

$$\chi_z \equiv \partial m / \partial H = \chi_\perp + H(\partial \chi_\perp / \partial H) \tag{2.3}$$

which is much easier than the direct diagrammatic calculation of $\chi_z(\mathbf{k})$.

The compact part $\hat{\Lambda}_{\alpha\alpha}(\mathbf{k})$ in (2.1) can be represented in the first order in $1/D$ by the diagram set from ref. 8 completed by additional diagrams for $H \neq 0$, which are selected according to the same rules. The general principle here is that the diagrams with multiple *irreducible* integrations over wavevectors (i.e., those that do not separate into products of independent simpler integrals) are small like the corresponding powers of $1/D$. Thus, in each order in $1/D$ the complexity of diagrams to be taken into account is restricted: in the zeroth order in $1/D$ (the spherical model) only the diagrams with the one-loop integration over the Brillouin zone survive, and in the first order in $1/D$ the double integrals over the Brillouin zone appear. The large number of diagrams in the case $H \neq 0$ necessitates, however, an improvement of the method, which consists in taking into account some diagrams *implicitly* with the subsequent solution of the corresponding equation for $\hat{\Lambda}_{\alpha\alpha}(\mathbf{k})$. All the diagrams which contribute to $\hat{\Lambda}_{\alpha\alpha}(\mathbf{k})$ in the first

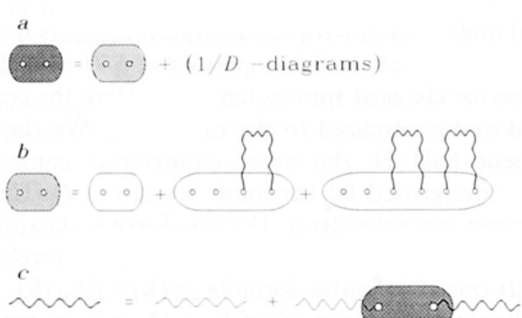

Fig. 1. (a) Diagrams for the compact part $\hat{\Lambda}_{\alpha\alpha}(\mathbf{k})$ (see also Fig. 2a); (b) block summation of transverse loops for the renormalized cumulant one-site two spin average $\tilde{\Lambda}_{\alpha\alpha}$; (c) Dyson equation for the renormalized transverse interaction.

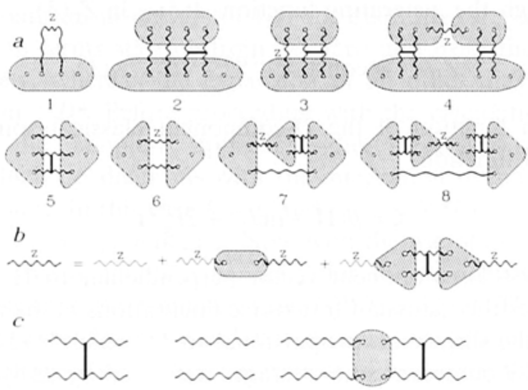

Fig. 2. (a) Additional $1/D$ diagrams for $\hat{\Lambda}_{\alpha\alpha}(\mathbf{k})$; (b) Dyson equation for the renormalized longitudinal interaction; (c) ladder equation for the four-spin correlation line.

order in $1/D$ are represented in Figs. 1 and 2. Note that the renormalized transverse interaction lines in Fig. 1 contain the unknown quantity $\hat{\Lambda}_{\alpha\alpha}(\mathbf{k})$ itself, which means implicitly accounting for the additional class of diagrams of types 1 and 2 in Fig. 3 of ref. 8. At $H=0$ from all the diagrams in Fig. 2a only the diagrams 1, 2, and 5 survive, and the last term in the Dyson equation for the longitudinal interaction line of Fig. 2b disappears. The wavevector dependence of $\hat{\Lambda}_{\alpha\alpha}(\mathbf{k})$ is due to the diagrams 5–8 in Fig. 2a. There is one more diagram 7′ that is analogous to 7 and is not represented in Fig. 2a to save space. Taking into account only the diagrams in Fig. 1 results in the self-consistent Gaussian approximation (SCGA), which describes rather well the thermodynamics of three dimensional ferromagnets.[10] The analytical form of $\hat{\Lambda}_{\alpha\alpha}(\mathbf{k})$ in Fig. 1 reads

$$\hat{\Lambda}_{\alpha\alpha}(\mathbf{k}) = \tilde{\Lambda}_{\alpha\alpha} + \hat{\Lambda}^{(1/D)}_{\alpha\alpha}(\mathbf{k}) \tag{2.4}$$

where $\hat{\Lambda}^{(1/D)}_{\alpha\alpha}(\mathbf{k})$ is the sum of the diagrams represented in Fig. 2a vanishing in the limit $D \to \infty$ (see appendix), and $\tilde{\Lambda}_{\alpha\alpha}$ is the renormalized two-spin cumulant average given by[8,10]

$$\tilde{\Lambda}_{\alpha\alpha} = \frac{1}{\pi^{(D-1)/2}} \int d^{D-1}r \exp(-r^2)\, \Lambda_{\alpha\alpha}(\zeta) \tag{2.5}$$

Here $\Lambda_{\alpha\alpha}$ is one of the cumulant spin averages of the general type[8]

$$\Lambda_{\alpha_1\alpha_2\cdots\alpha_p}(\xi) = \frac{\partial^p \Lambda(\xi)}{\partial\xi_{\alpha_1}\,\partial\xi_{\alpha_2}\cdots\partial\xi_{\alpha_p}} \tag{2.6}$$

obtained through the generating function $\Lambda(\xi) = \ln Z_0(\xi)$,

$$Z_0(\xi) = \text{const} \cdot \xi^{-(D/2-1)} I_{D/2-1}(\xi) \tag{2.7}$$

is the partition function of the D-component classical spin, $I_\nu(\xi)$ is the modified Bessel function,

$$\boldsymbol{\zeta} = \beta(\mathbf{H} + \mathbf{m}J_0) + 2l_\alpha^{1/2}\mathbf{r} \tag{2.8}$$

and $\mathbf{r}$ is the $(D-1)$-component vector perpendicular to $\mathbf{H}$. The last term in (2.8) describes the Gaussian transverse fluctuations of the molecular field $\mathbf{H} + \mathbf{m}J_0$ with the dispersion proportional to l_α, which leads to the renormalization of the cumulant spin averages described by (2.5) in the case of $\Lambda_{\alpha\alpha}$ and by analogous formulas for the other cumulants entering $\hat{\Lambda}_{\alpha\alpha}^{(1/D)}(\mathbf{k})$ (dashed ovals in Figs. 1 and 2). This renormalization results from the block summation of all the one-loop diagrams with the transverse interaction in Fig. 1b; the quantity l_α is given by the integral over the Brillouin zone

$$l_\alpha = \frac{1}{2} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{\beta J_{\mathbf{q}}}{1 - \hat{\Lambda}_{\alpha\alpha}(\mathbf{q})\, \beta J_{\mathbf{q}}} \tag{2.9}$$

where $v_0 = a_0^d$, a_0 is the atomic spacing, and d is the lattice dimensionality. The $(D-1)$-dimensional integral in (2.5) can be simplified by taking advantage of the symmetry with respect to the transverse variables and using the explicit form

$$\Lambda_{\alpha\alpha}(\boldsymbol{\xi}) = \frac{B(\xi)}{\xi}\left(1 - \frac{\xi_\alpha^2}{\xi^2}\right) + B'(\xi)\frac{\xi_\alpha^2}{\xi^2} \tag{2.10}$$

where

$$B(\xi) = \partial\Lambda(\xi)/\partial\xi = I_{D/2}(\xi)/I_{D/2-1}(\xi) \tag{2.11}$$

is the generalized Langevin function and $B'(\xi) \equiv dB/d\xi$. Making in (2.10) the substitution $\xi_\alpha^2 = \xi_r^2/(D-1)$ with $\xi_r \equiv 2l_\alpha^{1/2} r$ and in (2.5) the partial integration to get rid of B', one obtains

$$\tilde{\Lambda}_{\alpha\alpha} = \frac{2}{\Gamma((D+1)/2)} \int_0^\infty dr\, r^D \exp(-r^2) \frac{B(\zeta)}{\zeta}, \qquad \zeta = |\boldsymbol{\zeta}| \tag{2.12}$$

The formulas (2.4), (2.12), (2.8), and (2.9) determine implicitly the compact part of the spin–spin correlation function $\hat{\Lambda}_{\alpha\alpha}(\mathbf{k})$ entering the basic expression (2.1).

By the expansion in powers of $1/D$ the quantities $\tilde{\Lambda}_{\alpha\alpha}$ and $\hat{\Lambda}_{\alpha\alpha}^{(1/D)}(\mathbf{k})$ in (2.4) give rise to terms starting from the zero and from the first orders in $1/D$, respectively, whereas the expansion of all other diagrams neglected here starts from $1/D^2$. Before proceeding with the calculations we make a reference to the simplest approach—the mean-field approximation (MFA)—in which no diagrams with the integration over wavevectors are taken into account. In this case $l_\alpha \Rightarrow 0$ and $\zeta = \xi = \beta(H + mJ_0)$, and in (2.4), $\hat{\Lambda}_{\alpha\alpha}^{(1/D)}(\mathbf{k}) \Rightarrow 0$, $\tilde{\Lambda}_{\alpha\alpha} \Rightarrow \Lambda_{\alpha\alpha} \Rightarrow B(\xi)/\xi$. Now with the use of (2.1) and (2.2) one gets the Curie–Weiss equation $m = B(\xi)$ for the magnetization m, which yields the phase transition temperature $T_C^{\mathrm{MFA}} = |J_0|/D$. The latter has no physical significance for one- and two-dimensional magnets, but can be used as a temperature scale. It is convenient to introduce the dimensionless temperature $\theta \equiv T/T_C^{\mathrm{MFA}}$, magnetic field $h \equiv H/|J_0|$, and susceptibility $\tilde{\chi} \equiv |J_0|\chi$. Then the formulas (2.1) and (2.9) can be rewritten as

$$\tilde{\chi}_\perp(\mathbf{k}) = \frac{\hat{G}_\mathbf{k}}{1 - \nu \hat{G}_\mathbf{k} \lambda_\mathbf{k}}, \qquad \tilde{l}_\alpha \equiv \frac{l_\alpha}{D} = \frac{1}{2\theta} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{\lambda_\mathbf{q}}{1 - \nu \hat{G}_\mathbf{q} \lambda_\mathbf{q}} \tag{2.13}$$

where $\hat{G}_\mathbf{k} \equiv (D/\theta)\, \hat{\Lambda}_{\alpha\alpha}(\mathbf{k})$, $\nu = \pm 1$ for ferro- and antiferromagnets, and $\lambda_\mathbf{k} \equiv J_\mathbf{k}/J_0$. In the integral (2.12) the product $r^D \exp(-r^2)$ is at large D sharply peaked at $r = r_0 = (D/2)^{1/2}$, whereas $B(\zeta)/\zeta$ changes slowly with r. Using the expansion of $B(\xi)$, (2.11), for $D \gg 1$,[8] one can write

$$\frac{B(\zeta)}{\zeta} \cong \frac{2}{D}\left(g(x) + \frac{1}{D} \frac{x^2}{1+x^2} g^2(x) \right), \qquad g(x) = \frac{1}{1 + (1 + x^2)^{1/2}} \tag{2.14}$$

where

$$x = 2\zeta/D = x(\tilde{r}) = (4(h + \nu m)^2/\theta^2 + 8\tilde{l}_\alpha \tilde{r}^2)^{1/2}, \qquad \tilde{r} \equiv r/r_0 \tag{2.15}$$

and evaluate (2.12) by the pass method. In the first order in $1/D$ for $\tilde{G} \equiv (D/\theta)\, \tilde{\Lambda}_{\alpha\alpha}$ one gets

$$\tilde{G} = \frac{2}{\theta} \frac{1}{1 + [1 + 4(h + \nu m)^2/\theta^2 + 8\tilde{l}_\alpha]^{1/2}} + \frac{1}{D} \Delta^{(G)} \tag{2.16}$$

where

$$\Delta^{(G)} = \frac{2}{\theta}\left(\frac{x^2}{1+x^2} g^2(x) + \frac{1}{4}\frac{\partial g}{\partial \tilde{r}} + \frac{1}{4}\frac{\partial^2 g}{\partial \tilde{r}^2} \right)_{|\tilde{r}=1} \tag{2.17}$$

is the $1/D$ correction to the Gaussian integral (2.12) and the derivatives of g are calculated with the use of (2.14) and (2.15). The first term of (2.16) also contains the $1/D$ corrections due to the corresponding corrections to m and $\tilde{l}_\alpha$.

Before proceeding further with the $1/D$ expansion, we consider first the limiting case $D \to \infty$ corresponding to the spherical model.[13] In this case the quantities $\hat{\Lambda}_{\alpha\alpha}(\mathbf{k})$ in (2.4) and $\Delta^{(G)}$ in (2.17) can be neglected, and one comes to the closed system of equations

$$G = \frac{2}{\theta} \frac{1}{1 + [1 + 4(h + \nu m_0)^2/\theta^2 + 8\tilde{l}_{\alpha 0}]^{1/2}} \tag{2.18}$$

and

$$\tilde{l}_{\alpha 0} = \frac{P(G) - 1}{2\theta G}; \qquad P(G) = v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{1 - G\lambda_{\mathbf{q}}} \tag{2.19}$$

Here for the square lattice with n.n. interaction the lattice integral $P(G)$ is given by $P(G) = (2/\pi)\,\mathbf{K}(k)$ with $k = G$, $\mathbf{K}(k)$ being the elliptic integral of the first kind, and for the linear chain $P(G) = 1/(1 - G^2)^{1/2}$. For bipartite lattices considered here the integral $P(G)$ is the same for ferro- and antiferromagnets and independent of the sign of G. For this reason the sign factor ν is dropped in the definition of $P(G)$, (2.19). Note that for one- and two-dimensional systems $P(G)$ diverges for $G \to 1$, which is the reason for the absence of long-range order. Eliminating now $\tilde{l}_{\alpha 0}$ from (2.18) and (2.19) and using (2.2) in the form

$$\frac{m_0}{h} = \frac{G}{1 - \nu G} \tag{2.20}$$

one comes to the equation of state of the spherical model:

$$\theta G P(G) = 1 - m_0^2 \tag{2.21}$$

which for $h \neq 0$ should be solved together with (2.20) (in the general case numerically). For low-dimensional antiferromagnets ($\nu = -1$) at low temperatures ($\theta \ll 1$) in the field region where $m_0^2 < 1$ Eq. (2.21) requires $G \cong 1$ and, correspondingly, $P(G) \gg 1$. For the square-lattice model for $P(G)$ and its derivative $P' \equiv dP/dG$ this implies

$$\begin{aligned} P(G) &\cong \frac{1}{\pi} \ln\left(\frac{8}{1 - G}\right) \cong \frac{1 - m_0^2}{\theta} \\ P'(G) &\cong \frac{1}{\pi} \frac{1}{1 - G} \cong \frac{1}{8\pi} \exp\left[\frac{\pi(1 - m_0^2)}{\theta}\right] \end{aligned} \tag{2.22}$$

i.e., the deviation of G from unity is exponentially small:

$$G \cong 1 - 8 \exp\left[-\frac{\pi(1 - m_0^2)}{\theta}\right] \tag{2.23}$$

For the linear chain model the corresponding result reads $1-G\cong \theta^2/[2(1-m_0^2)^2]$. Now with the help of (2.20) one gets for the magnetization $m_0\cong h/2$ with only exponentially small corrections in the two-dimensional case due to (2.23). The latter is valid up to the magnetization saturation point $h=2$ (i.e., $H=2\,|J_0|$), which corresponds to the spin-flip field of three-dimensional antiferromagnets. For the fields $h>2$ according to (2.20) and (2.21), $m_0\cong 1$ and $G<1$:

$$m_0\cong 1-(\theta/2)\,GP(G); \qquad G\cong 1/(h-1) \tag{2.24}$$

In the zero-field case the antiferromagnetic susceptibility $\tilde{\chi}(\theta)=\tilde{\chi}_{\perp}=\tilde{\chi}_{z}=G/(1+G)$ monotonically decreases from the value 1/2 at $\theta=0$ to 0 at $\theta\to\infty$, i.e., the spherical model does not describe the maximum of the antiferromagnetic susceptibility at $\theta\lesssim 1$.

Now, returning to the $1/D$ expansion, one can express the $1/D$-correction term $\Delta^{(G)}$ in (2.16) through the variables of the spherical approximation:

$$\Delta^{(G)}=2G\left[\frac{y-1}{2y-1}-\frac{P-1}{2(2y-1)^2}-\frac{(P-1)^2\,(3y-1)}{(2y-1)^3}\right], \qquad y\equiv\frac{1}{\theta G} \tag{2.25}$$

and represent the unknown quantities m, $\tilde{l}_{\alpha}$, and $\hat{G}_{\mathbf{k}}$ [see (2.13)] in the form

$$\begin{aligned} m &\cong m_0+m_1/D \\ \tilde{l}_{\alpha} &\cong \tilde{l}_{\alpha 0}+\tilde{l}_{\alpha 1}/D \\ \hat{G}_{\mathbf{k}} &\cong G+\Delta G_{\mathbf{k}}/D \end{aligned} \tag{2.26}$$

Here the corrections m_1 and $\tilde{l}_{\alpha 1}$ can be expressed through $\Delta\hat{G}_{\mathbf{k}}$ with the use of (2.13) and the relation $m/h=\tilde{\chi}_{\perp}(0)=\hat{G}_0/(1-v\hat{G}_0)$, which results in

$$\frac{m_1}{h}=\frac{\Delta G_0}{(1-vG)^2} \tag{2.27}$$

and

$$\tilde{l}_{\alpha 1}=\frac{1}{2\theta}\,v_0\int\frac{d\mathbf{q}}{(2\pi)^d}\frac{\lambda_{\mathbf{q}}^2\,\Delta G_{\mathbf{q}}}{(1-vG\lambda_{\mathbf{q}})^2} \tag{2.28}$$

Expanding now the first term of (2.16) up to the first order in m_1 and $\tilde{l}_{\alpha 1}$, one comes to the $1/D$ part of Eq. (2.4):

$$\Delta G_{\mathbf{k}}+\frac{m_0}{vh}\frac{2m_0^2 y}{2y-1}\,\Delta G_0+\frac{G^2}{2y-1}\,v_0\int\frac{d\mathbf{q}}{(2\pi)^d}\frac{\lambda_{\mathbf{q}}^2\,\Delta G_{\mathbf{q}}}{(1-vG\lambda_{\mathbf{q}})^2}=\Delta^{(G)}+\Delta_{\mathbf{k}}^{(1/D)} \tag{2.29}$$

where the quantity $\Delta_{\mathbf{k}}^{(1/D)}$ is the part of $D \cdot (D/\theta)\, \hat{\Lambda}_{\alpha\alpha}^{(1/D)}(\mathbf{k})$ nonvanishing in the limit $D \to \infty$ (see appendix). The solution of the integral equation (2.29) has the form

$$\Delta G_{\mathbf{k}} = \Delta G_0 + M_{\mathbf{k}} \tag{2.30}$$

where

$$M_{\mathbf{k}} = \Delta_{\mathbf{k}}^{(1/D)} - \Delta_0^{(1/D)} \tag{2.31}$$

and ΔG_0 is given by

$$\begin{aligned}\Delta G_0 = &\left\{(2y-1)(\Delta^{(G)} + \Delta_0^{(1/D)}) - G^2 v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{\lambda_{\mathbf{q}}^2 M_{\mathbf{q}}}{(1 - vG\lambda_{\mathbf{q}})^2}\right\} \\ &\times \left\{GP' + P(G) + \frac{2m_0^2 y}{1 - vG}\right\}^{-1}\end{aligned} \tag{2.32}$$

Now, calculating the quantity $\Delta_{\mathbf{k}}^{(1/D)}$ (see appendix) and introducing the function

$$r_{\mathbf{q}} = v_0 \int \frac{d\mathbf{p}}{(2\pi)^d} g_{\mathbf{p}} g_{\mathbf{p}-\mathbf{q}}, \qquad g_{\mathbf{p}} \equiv \frac{1}{1 - vG\lambda_{\mathbf{p}}} \tag{2.33}$$

one arrives after numerous cancellations at the final results for ΔG_0 and $M_{\mathbf{k}}$:

$$\begin{aligned}\Delta G_0 = 2G &\left\{1 - [GP' + P + 2m_0^2 y)]\, v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{g_{\mathbf{q}}}{\tilde{r}_{\mathbf{q}}} + m_0^2 y v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{g_{\mathbf{q}}^2}{\tilde{r}_{\mathbf{q}}}\right. \\ &\left. + \frac{G}{2} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{r'_{\mathbf{q}}}{\tilde{r}_{\mathbf{q}}}\right\}\left\{GP' + P + \frac{2m_0^2 y}{1 - vG}\right\}^{-1}\end{aligned} \tag{2.34}$$

and

$$M_{\mathbf{k}} = 2Gv_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{g_{\mathbf{q}} - g_{\mathbf{q}-\mathbf{k}}}{\tilde{r}_{\mathbf{q}}} \tag{2.35}$$

where $\tilde{r}_{\mathbf{q}} \equiv r_{\mathbf{q}} + 2m_0^2 y g_{\mathbf{q}}$ and $r'_{\mathbf{q}} \equiv \partial r_{\mathbf{q}}/\partial G$. Similar results obtained earlier[14,15] by another method for the particular case of ferromagnets were used for the investigation of the phase transition in three dimensions. It is interesting to note that the function $r_{\mathbf{q}}$ of (2.33) is [like $P(G)$] identical for ferro- and antiferromagnets and has a singularity at $G \to 1$ and $\mathbf{q} \to 0$. In contrast, the quantity $\tilde{r}_{\mathbf{q}}$ entering (2.34) and (2.35) has for antiferromagnets one more singularity at $G \to 1$ and $\mathbf{q} \to \mathbf{b}$ ($\mathbf{b}$ is the inverse lattice vector corresponding to the corner of the Brillouin zone) due to $g_{\mathbf{q}}$, which disappears, however, in zero magnetic field ($m_0 = 0$). This is a formal mechanism responsible for the singular behavior of the susceptibility $\chi(H, T)$ of low-dimensional antiferromagnets in the limit H, $T \to 0$ discussed in the Introduction.

Before proceeding with the application of the results obtained to concrete systems, it is worth noting some general properties of the $\mathbf{k}$-dependent spin–spin correlation function [see (2.1) and (2.13)] that can essentially simplify the consideration in the low-temperature range. In particular, for two-dimensional ferromagnets in the spherical limit the quantity G in (2.23) is exponentially close to unity at $h = 0$ and low temperatures, which implies exponentially small gap in the spin-wave spectrum. Since this property cannot change upon taking into account $1/D$ corrections, the quantity ΔG_0 in (2.30) also should be exponentially small. This is physically clear and can be confirmed by the direct analysis[8] of the results obtained. On the other hand, the $\mathbf{k}$-dependent contribution $M_{\mathbf{k}}$ in (2.30) does not have to be exponentially small at low temperatures and $\mathbf{k} \neq 0$. In fact, the value $M_{\mathbf{b}}$ determines the $1/D$ correction to the antiferromagnetic (staggered) susceptibility of a ferromagnetic model ($J > 0$) in zero magnetic field,[8] which can be expanded in powers of $\theta \ll 1$. Thus, by calculation of such quantities of two-dimensional magnetic systems at low temperatures, which are not exponentially small, one can use only the quantity $M_{\mathbf{k}}$ of (2.35), being much simpler than the expression for ΔG_0, (2.34). This means that only $\mathbf{k}$-dependent diagrams for the compact part of the spin–spin correlation function $\hat{\Lambda}_{\alpha\alpha}(\mathbf{k})$ should be taken into account in the low-temperature range, which is a clear advantage of the diagrammatic $1/D$ expansion in comparison with the earlier version.[14,15] The considerations above can be extended also to two-dimensional antiferromagnets ($J < 0$) at low temperatures in the field region $h < 2$ ($H < 2\,|J_0|$), where the magnon gap is also exponentially small. Here the quantity $\hat{G}_{\mathbf{b}}$ in (2.13) should be exponentially close to unity in all orders in $1/D$. Consequently, the quantity ΔG_0 contributing to the magnetization and susceptibility of an antiferromagnet [see (2.27)] is given according to (2.30) by $\Delta G_0 = \Delta G_{\mathbf{b}} - M_{\mathbf{b}} \cong -M_{\mathbf{b}}$. In the next sections we apply the results of the first order in $1/D$ obtained above to the analysis of the equation of state $m(H, T)$ of one- and two-dimensional classical antiferromagnets.

3. THE LINEAR CHAIN CLASSICAL SPIN MODEL

For the linear chain model $\lambda_k \equiv J_k/J_0 = \cos(a_0 k)$, and the integrals (2.33) and (2.34) can be calculated analytically. One gets $r_q = 2P(G)/(2 - G^2 - G^2\lambda_q)$ with $P(G) = 1/(1 - G^2)^{1/2}$ and

$$\Delta G_0 = \frac{2G(1-m_0^2)(1-G^2)^{3/2}}{1+m_0^2+2m_0^2 \nu G} \left\{ 1 + \frac{3}{2} P(G) - \frac{5+3m_0^2-2m_0^2 G^2}{2[1+m_0^2(1-G^2)]} F \right.$$
$$\left. + \frac{1}{2} \frac{1-F}{1-m_0^2(1-\nu G)} \left[\frac{1-m_0^2}{1-G^2}(1-\nu G) - 3(1-m_0^2) - 2\nu G m_0^2 \right] \right\} \tag{3.1}$$

where

$$F = \frac{1+m_0^2(1-G^2)}{[(1+m_0^2)^2(1-G^2)+2G^2 m_0^2(1-m_0^2)(1-\nu G)]^{1/2}} \tag{3.2}$$

The magnetization m_0 and the parameter G of the spherical model in (3.1) and (3.2) are given by the solution of (2.20) and (2.21). It can be shown that in zero magnetic field the results for the susceptibility χ are equivalent to those obtained by the expansion of the exact solution[12] up to the first order in $1/D$. In the low-field and low-temperature limit h, $\theta \ll 1$ one has $m_0 \cong h/2 \ll 1$, $G \cong 1-\theta^2$, and hence $F \cong (h^2+\theta^2)^{-1/2} \gg 1$. Taking into account the leading contribution to ΔG_0 given by the first term in square brackets, one gets with the use of (2.26) and (2.27) the result

$$\tilde{\chi}_\perp = \frac{m}{h} = \frac{1}{2}\left[1 + \frac{1}{D}\left(-\frac{\theta}{(h^2+\theta^2)^{1/2}} + \theta + O(\theta^2) \right) \right] \tag{3.3}$$

It can be seen that for $h=0$ the susceptibility $\tilde{\chi}_\perp$ decreases with lowering temperature due to the term θ in (3.3) and attains the value $\tilde{\chi}_\perp = \frac{1}{2}(1-1/D)$ at $\theta=0$. If $h \neq 0$, then at $\theta = h^{2/3}$ the value of $\tilde{\chi}_\perp$ attains a minimum and then rises to 1/2 at $\theta=0$. Note that the singular term in (3.3) becomes of order unity at $\theta \sim h \ll h^{2/3}$, which is one more characteristic temperature. Such a qualitative behavior of the susceptibility of a low-dimensional classical antiferromagnet is in accord with the physical considerations in the Introduction. The longitudinal susceptibility $\tilde{\chi}_z$ calculated with the help of (2.3) and (3.3) has the form

$$\tilde{\chi}_z = \frac{\partial m}{\partial h} = \frac{1}{2}\left[1 + \frac{1}{D}\left(-\frac{\theta^3}{(h^2+\theta^2)^{3/2}} + \theta + O(\theta^2) \right) \right] \tag{3.4}$$

This expression has a minimum at $\theta \cong 3^{1/3} h^{2/3} \gg h$ and a maximum at $\theta \cong 3^{-1/2} h^{3/2} \ll h$ (the third characteristic temperature) where $\tilde{\chi}_z \cong 1/2 + (2/D)(h/3)^{3/2} > 1/2$. The susceptibilities $\tilde{\chi}_\perp(h, \theta)$ and $\tilde{\chi}_z(h, \theta)$ are represented as functions of temperature for some field values in Figs. 3 and 4.

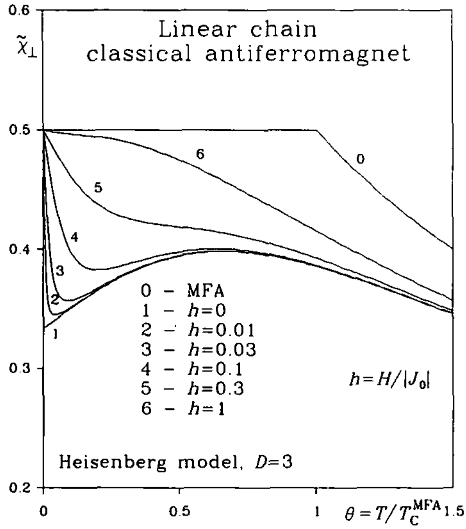

Fig. 3. Temperature dependence of the transverse susceptibility $\tilde{\chi}_\perp = m/h$ of the l.c. Heisenberg antiferromagnet for different magnetic fields in the first order in $1/D$.

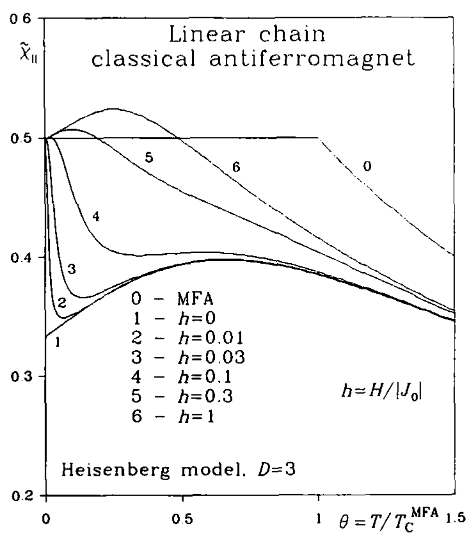

Fig. 4. Temperature dependence of the longitudinal susceptibility $\tilde{\chi}_z = \partial m/\partial h$.

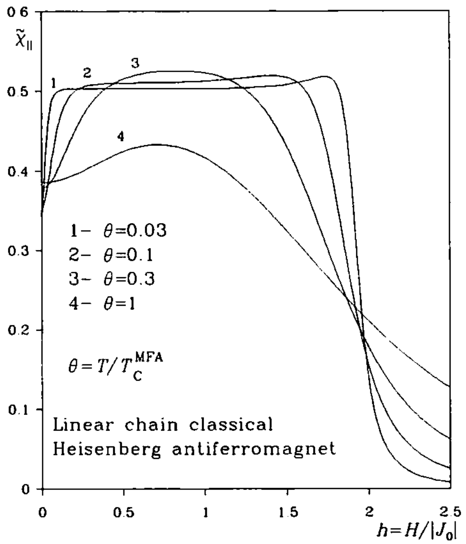

Fig. 5. Field dependence of the longitudinal susceptibility of the l.c. classical Heisenberg antiferromagnet for different temperatures.

An interesting feature of the susceptibility $\tilde{\chi}_z$ manifests itself in the $1/D$ approximation in the low-temperature limit ($\theta \ll 1$) in the vicinity of the magnetization saturation point $h = 2$ (i.e., $H = H_c \equiv 2\,|J_0|$). In the spherical limit, adopting $m_0 = 1 - \delta m_0$ and $G = 1 - \delta G$ with δm_0, $\delta G \ll 1$, one can simplify Eqs. (2.20) and (2.21) for the linear chain antiferromagnet to $\theta/(2\delta G)^{1/2} = 2\delta m_0$ and $2\delta m_0 - \delta G = 2 - h$, which results in the following equation for δm_0 in the scaled form:

$$x - 1/(16x^2) = x_0; \qquad x \equiv \delta m_0/\theta^{2/3}, \qquad x_0 \equiv (2-h)/(2\theta^{2/3}) \tag{3.5}$$

This equation describes the temperature-induced rounding of the transition between the dependences $m_0 \cong h/2$ and $m_0 \cong 1$ in the small field interval $|2-h| \sim \theta^{2/3}$. Now, the $1/D$ correction m_1 determined for $|2-h|$, $\theta \ll 1$ from (2.27) and (3.1), (3.2) has the form

$$m_1 \cong \delta m_0(3 - 3Y^{-1/2} + Y^{1/2})/Y, \qquad Y \equiv 1 + \delta m_0/\delta G = 1 + 8x^3 \tag{3.6}$$

where x is the solution of (3.5). In the limiting cases one gets from (3.7) and (3.6) for the magnetization $m = m_0 + m_1/D$ the following results

$$m \cong \begin{cases} \dfrac{h}{2} + \dfrac{1}{D}\dfrac{\theta}{2(2-h)^{1/2}} - \dfrac{\theta^2}{[2(2-h)]^2}, & \theta^{2/3} \ll 2-h \ll 1 \\ 1 - \left(\dfrac{\theta}{2}\right)^{2/3}\left[1 - \dfrac{1}{D}\left(\dfrac{2}{3}\right)^{1/2}(6^{1/2}-1)\right], & h=2 \\ 1 - \dfrac{\theta}{[8(h-2)]^{1/2}}\left(1-\dfrac{1}{D}\right), & \theta^{2/3} \ll h-2 \ll 1 \end{cases} \tag{3.7}$$

It can be seen from (3.7) that in the field region below the saturation point $h=2$ the temperature-dependent correction to m is positive. Accordingly, the susceptibility $\tilde{\chi}_z = \partial m/\partial h$ exceeds 1/2 in this region, but the effect is not great. With the use of (3.6) one can show that for $D=3$, $\tilde{\chi}_{z,\max} = 0.518$ at $2-h = 2.80 \cdot \theta^{2/3}$. The field dependence of the normalized susceptibility $\tilde{\chi}_z$ of the one-dimensional classical antiferromagnet is represented for different temperatures in Fig. 5. It is interesting to note that a qualitatively similar field dependence of the susceptibility with a logarithmic singularity at small fields was found in ref. 16 for the *quantum* linear chain Heisenberg antiferromagnet with $S=1/2$ at $T=0$. There are no physical comments to this effect in ref. 16, but it seems now rather plausible that the origin of this low-field singularity of a quantum antiferromagnet is also the orientation of sublattices perpendicular to the field (see Introduction). For $H \to 0$, quantum fluctuations destroy this orientation, and the susceptibility diminishes.

4. THE SQUARE-LATTICE CLASSICAL ANTIFERROMAGNET

In two dimensions the integrals in ΔG_0 in (2.34) cannot be calculated analytically. For the convenience of the analysis at low temperatures and numerical calculations we introduce instead of the strongly singular $r_{\mathbf{q}}$ of (2.33) the weak-singular function[8]

$$\psi_{\mathbf{q}} \equiv \frac{1}{G} v_0 \int \frac{d\mathbf{p}}{(2\pi)^d} \frac{\lambda_{\mathbf{q}} - G\lambda_{\mathbf{p}}\lambda_{\mathbf{p}-\mathbf{q}}}{(1-G\lambda_{\mathbf{p}})(1-G\lambda_{\mathbf{p}-\mathbf{q}})} = \frac{1}{G^2}\left[2P(G) - 1 - (1-G\lambda_{\mathbf{q}})\, r_{\mathbf{q}}\right] \tag{4.1}$$

in which the divergence of the integrand at $\mathbf{p}=0$ and $\mathbf{p}=\mathbf{q}$ at $G \cong 1$ are partially compensated by the nullification of the numerator. In the long-wavelength region the function $\psi_{\mathbf{q}}$ has the form[8]

$$\psi_{\mathbf{q}} \cong \begin{cases} \dfrac{2}{\pi}\ln\dfrac{8}{1-G} - 1 - \dfrac{1}{\pi}, & x \equiv (a_0 q)^2 \ll 1-G \\ \dfrac{2}{\pi}\ln\dfrac{8}{x}, & 1-G \ll x \ll 1 \end{cases} \tag{4.2}$$

and its derivative $\psi'_{\mathbf{q}} \equiv \partial\psi_{\mathbf{q}}/\partial G$ is given by

$$\psi'_{\mathbf{q}} \cong \begin{cases} \dfrac{2}{\pi}\dfrac{1}{1-G}, & x \ll 1-G \\ -\dfrac{2}{\pi x}\ln\dfrac{x}{1-G}, & 1-G \ll x \ll 1 \end{cases} \tag{4.3}$$

At low temperatures in the corners of the Brillouin zone $\psi_{\mathbf{b}} = -1 + O(1-G)$. In terms of $\psi_{\mathbf{q}}$ the function ΔG_0 of (2.34) can be written as

$$\begin{aligned} \Delta G_0 = 2G \Bigg\{ & (1-\nu)\frac{GP' + P + 3m_0^2 y}{2y-1} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{1-a\psi_{\mathbf{q}}} \frac{G\lambda_{\mathbf{q}}}{1+\bar{G}_{\mathbf{q}}\lambda_{\mathbf{q}}} \\ & + \frac{G}{2} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{a\psi'_{\mathbf{q}}}{1-a\psi_{\mathbf{q}}} \frac{1+G\lambda_{\mathbf{q}}}{1+\bar{G}_{\mathbf{q}}\lambda_{\mathbf{q}}} \\ & - \frac{3y-1}{2y-1} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{1-a\psi_{\mathbf{q}}} \frac{1+G\lambda_{\mathbf{q}}}{1+\bar{G}_{\mathbf{q}}\lambda_{\mathbf{q}}} \\ & + \frac{3}{2} + P(G) - \frac{1}{2} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{1+\bar{G}_{\mathbf{q}}\lambda_{\mathbf{q}}} \Bigg\} \\ & \times \left\{ GP' + P(G) + \frac{2m_0^2 y}{1-\nu G} \right\}^{-1} \end{aligned} \tag{4.4}$$

where $y \equiv 1/(\theta G)$, $a \equiv G^2/(2y-1)$, and

$$\bar{G}_{\mathbf{q}} \equiv G\left[1 - \frac{2(1-\nu)\, m_0^2 y}{(2y-1)(1-a\psi_{\mathbf{q}})}\right] \tag{4.5}$$

For the quantity $M_{\mathbf{k}}$ of (2.35) one gets in a similar way

$$M_{\mathbf{k}} = \frac{2G}{2y-1} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{(1-G^2\lambda_{\mathbf{q}}^2)(g_{\mathbf{q}} - g_{\mathbf{q}-\mathbf{k}})}{(1-a\psi_{\mathbf{q}})(1+\bar{G}_{\mathbf{q}}\lambda_{\mathbf{q}})} \tag{4.6}$$

Putting $\mathbf{k} = \mathbf{b}$ in (4.6) in the antiferromagnetic case ($\nu = -1$) and taking into account only the exponentially great terms with $P'(G)$ in ΔG_0, (4.4), at low temperatures in the field range $h < 2$, one arrives at the result

$$\Delta G_0 \cong -M_{\mathbf{b}} = 4av_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{1-a\psi_{\mathbf{q}}} \frac{\lambda_{\mathbf{q}}}{1+\bar{G}_{\mathbf{q}}\lambda_{\mathbf{q}}} \tag{4.7}$$

which confirms the conjecture made at the end of Section 2. Since at $\theta \ll 1$ one has $a \cong (\theta/2)/(1-\theta/2) \ll 1$ and due to (4.2) the functions $\psi''_{\mathbf{q}}$ are integrable, one can expand (4.7) in powers of $a\psi_{\mathbf{q}}$ and then of θ to get the development of the $1/D$ correction to the magnetization m_1 in (2.27) at low temperatures. In the lowest order in θ one gets

$$m_1 = \frac{h\theta}{2} \frac{1 - P(\bar{G})}{1 - 2m_0^2} \tag{4.8}$$

where $\bar{G} \cong G(1 - 2m_0^2)$ and $m_0 \cong h/2$. This correction is negative for $h < 2^{1/2}$ and positive for $2^{1/2} < h \lesssim 2$. In the case $h \neq 0$ the quantity $1 - \bar{G}$ can be interpreted as proportional to the field-induced gap of the out-of-plane spin waves, which makes the lattice integral $P(\bar{G})$ in (4.8) not divergent at low temperatures. A more detailed physical interpretation of the spin-wave dynamics in low-dimensional magnets requires, however, the dynamical generalization of the diagram technique used here. In the small-field region, where $\bar{G} \cong 1$, with the help of (2.26) and (2.2) in the first order in $1/D$ one gets

$$\tilde{\chi}_\perp = \frac{m}{h} = \frac{1}{2}\left[1 + \frac{1}{D}\left(-\frac{\theta}{\pi}\ln\frac{8}{1 - G + h^2/2} + \theta + O(\theta^2)\right)\right] \tag{4.9}$$

where G is given by (2.23). In the case $h = 0$ the ln term in (4.9) is identically equal to -1, and $\tilde{\chi}_\perp \to \frac{1}{2}(1 - 1/D)$ in the limit $\theta \to 0$. For arbitrary small field $h \neq 0$ this term goes to zero with $\theta \to 0$, and $\tilde{\chi}_\perp \to 1/2$. The transition to the regime where the magnetic field exerts an influence on the susceptibility of a two dimensional antiferromagnet is sharp due to the strong exponential temperature dependence of G, (2.23), and occur at the temperature

$$\theta \cong \theta^* = \frac{\pi}{2\ln(4/h)} \tag{4.10}$$

Note that for $h \ll 1$ the value of θ^* is much larger than the corresponding characteristic temperatures in the one-dimensional case [see (3.3) and (3.4)]. The longitudinal susceptibility of the s.l. classical antiferromagnet has the form

$$\tilde{\chi}_z = \frac{1}{2}\left[1 + \frac{1}{D}\left(-\frac{\theta}{\pi}\ln\frac{8}{1 - G + h^2/2} + \frac{\theta}{\pi}\frac{h^2}{1 - G + h^2/2} + \theta + O(\theta^2)\right)\right] \tag{4.11}$$

where the additional term in comparison to (4.9) is not very essential at low fields, in contrast to the one-dimensional case [see (3.4)]. The temperature dependences of $\tilde{\chi}_\perp$ and $\tilde{\chi}_z$ in a magnetic field obtained by the

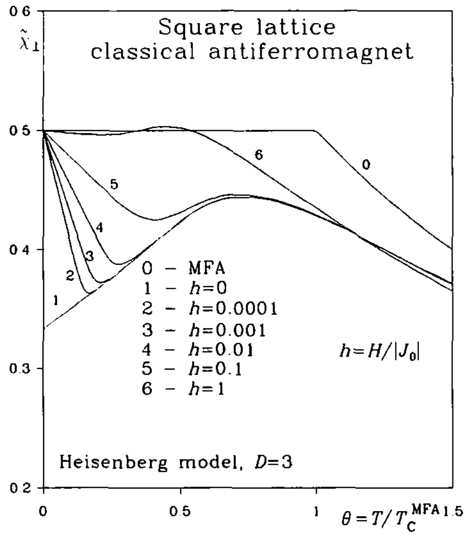

Fig. 6. Temperature dependence of the transverse susceptibility of the s.l. classical Heisenberg antiferromagnet for different magnetic fields in the first order in $1/D$.

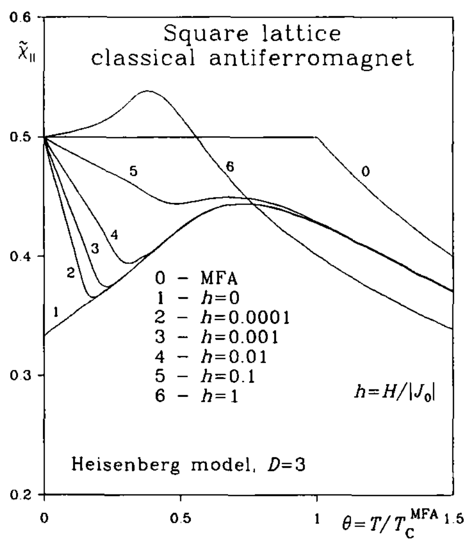

Fig. 7. Temperature dependence of the longitudinal susceptibility.

numerical solution of Eqs. (2.20) and (2.21) and the numerical integration in (4.4) are represented in Figs. 6 and 7.

The $1/D$ correction to the magnetization (4.8) diverges upon approaching the magnetization saturation point $h=2$, since here $\bar{G} \to -1$ and $P(\bar{G}) \to \infty$. In fact, the formula (4.8) becomes inadequate in this region, because in (4.4) the integral with $\psi'_{\mathbf{q}}$ [see (4.3)] becomes for $\bar{G} \cong -1$ comparable with the one with $P'(\bar{G})$ due to the large long-wavelength contribution, and the quantity m_1 remains finite. In contrast, the quantity $a\psi_{\mathbf{q}}$ in the denominators in (4.4) and (4.5) can be neglected in the whole field region, since in the low-temperature range $a\psi_0 \cong \theta P(G) \cong 1-m_0^2$. In the field region above the saturation point ($h>2$) at low temperatures $m_0 \cong 1$ and in (4.5) $\bar{G}_{\mathbf{q}} \cong -G$. Neglecting the terms $\theta P' \ll 1$ in ΔG_0, (4.4), with the use of (2.24) for the total magnetization $m = m_0 + m_1/D$, one gets

$$m \cong 1 - \frac{\theta}{2}\left(1-\frac{1}{D}\right) GP(G), \qquad G \cong \frac{1}{h-1} \qquad (h>2, \theta \ll 1) \tag{4.12}$$

This result is the exact expression for the leading correction to the magnetization of a classical antiferromagnet in the spin-flip phase ($H > 2\,|J_0|$) in the low-temperature limit, which can be obtained independently with the help of the lowest order spin-wave theory. In the framework of the diagram technique for classical spin systems used here this corresponds to taking into account only the simplest diagram for the magnetization m with one integration over the Brillouin zone (i.e., the one analogous to the second diagram in Fig. 1b). The derivation of the formula (4.12) is trivial, because the ground state of the system has no spontaneous symmetry breaking and the magnon spectrum has a gap. However, with the approach to $h=2$ in (4.12) $G \to 1$ and for low-dimensional systems the spin-wave correction to m diverges. In the region $h \lesssim 2$ the situation becomes complicated, and to obtain finite results for the thermodynamic quantities one has to take into account an infinite series of diagrams, which is exemplified by the $1/D$ expansion described above. With the help of (4.8) and (4.12) one can write down the expressions for the magnetization of the two-dimensional antiferromagnet on both sides of the magnetization saturation point $h=2$ excluding a small intermediate region:

$$m \cong \begin{cases} \dfrac{h}{2} + \dfrac{\theta}{\pi D} \ln \dfrac{4}{2-h}, & \theta \ln \dfrac{1}{\theta} \ll 2-h \ll 1 \\[2ex] 1 - \dfrac{\theta}{2\pi}\left(1-\dfrac{1}{D}\right) \ln \dfrac{8}{h-2}, & \theta \ln \dfrac{1}{\theta} \ll h-2 \ll 1 \end{cases} \tag{4.13}$$

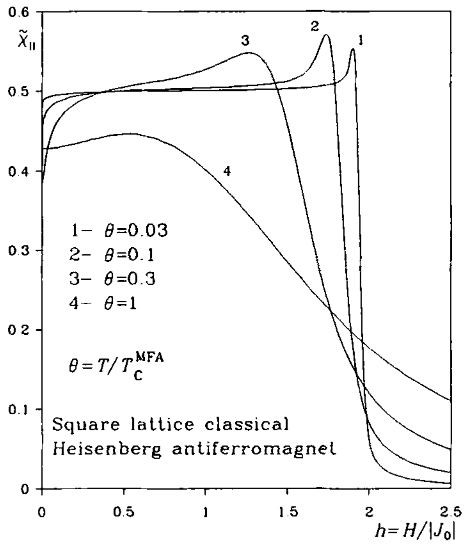

Fig. 8. Field dependence of the longitudinal susceptibility of the s.l. classical Heisenberg antiferromagnet for different temperatures.

The latter results are analogous to those for the linear chain model (3.7), which could also be obtained in the same way as here. The normalized susceptibility $\tilde{\chi}_z$ of the two-dimension antiferromagnet is greater then 1/2 below the saturation point $h=2$, too, and the maximal value of $\tilde{\chi}_z$ is greater than that for the linear chain (see Fig. 8). The latter can be explained by the fact that for a square lattice there is no competing *negative* contribution to m of the zeroth order in $1/D$, as is the case for the linear chain [see (3.7)].

5. DISCUSSION

In this article the $1/D$ expansion of the physical quantities of low-dimensional classical D-vector models in the whole range of temperatures and magnetic fields was developed, the results obtained being valid for both ferro- and antiferromagnets. For the calculation of the susceptibility and the field-induced magnetization of ferromagnets at low temperatures the method is, however, not very efficient, because these quantities are singular at $T \to 0$. In ref. 8 it was shown that at low temperatures the $1/D$ correction to the susceptibility of a two-dimensional ferromagnet becomes greater than its value in the zeroth order in $1/D$, which means that D enters the argument of the exponentially great expression for $\chi \propto 1/(1-\hat{G}_0)$ [see

(2.22)]. This is in accord with the results of the RG approach of ref. 17, but does not allow one to obtain accurate results in the framework of the $1/D$ expansion.

On the other hand, the $1/D$ expansion proves to be a very good instrument for the description of *noncritical* characteristics of low-dimensional magnets, such as the magnetization and susceptibility of antiferromagnets and the energy and heat capacity of both ferro- and antiferromagnets. For the latter the zero-field results (identical in both cases) were obtained in ref. 8, and their generalization for the case with a magnetic field with the use of the methods developed here offers no difficulties. The most remarkable feature of the results obtained with the help of the $1/D$ expansion is that they describe the maximum in the temperature dependence of the zero-field antiferromagnetic susceptibility and its singular behavior at $H, T \to 0$. The former is the result of taking into account the diagrams with *double* integrations over the Brillouin zone, which was not done in any of the preceding theories. This means allowing for the wavevector dependence of the compact part of the spin–spin correlation function (2.1) as well, or, in the other words, going beyond the Ornstein–Zernike form for $\chi_{\mathbf{k}}$.

An intriguing property of the $1/D$ expansion is that it leads to the exact results for the *noncritical* characteristics of low-dimensional magnets at low temperatures. All the examples considered up to now suggest that for $H = 0$ the coefficients in the expansions of the noncritical quantities in powers of θ are polynomials in $1/D$ (see ref. 8). If this is true, then the hitherto unavailable low-temperature expansions of these quantities can be obtained with the help of the $1/D$ expansion! Further, this should imply that there is some method of derivation of these low-temperature expansions without using the $1/D$ expansion. A search for such a method is planned for the nearest future.

It would be very interesting to compare the results of the $1/D$ expansion with results obtained by other methods. In particular, for the energy of a square-lattice classical Heisenberg magnet, MC simulations were made by Shenker and Tobochnik[11] (see the comparison in ref. 8), but the antiferromagnetic susceptibility was simulated by various researches only for a quantum model with $S = 1/2$. As concerns the two-dimensional model with $D = 2$, the $1/D$ expansion cannot, naturally, describe the Kosterlitz–Thouless transition, which occurs in this system. But one can expect that the general features of the temperature dependence of the antiferromagnetic susceptibility in magnetic fields described by the $1/D$ expansion are inherent for this model, too. Moreover, in a magnetic field the behavior of the antiferromagnetic model with $D = 2$ should simplify, because the magnetic field lifts the spontaneous symmetry breaking and induces the gap of spin fluctuations. In this case at low temperatures it is enough to take into account only the lowest order diagram of the spin-wave theory, which is naturally contained in the $1/D$ expansion in the first order in $1/D$ [see (4.12)]. It should be stressed that for the model with $D=2$ the effects in the temperature and field dependences of the antiferromagnetic susceptibility discussed in this paper should show up most strongly. For the Heisenberg ($D=3$) antiferromagnet the behavior in the magnetic field can be more complicated than for $D=2$. As we have seen, at low temperatures even a small magnetic field forces the spins to lie perpendicular to it. This decreases the effective number of spin components from $D=3$ to $D=2$ and should lead to the Kosterlitz–Thouless transition with disappearance of the gap. But it should not change essentially the results for the susceptibility in this region, since in the expression for the $1/D$ correction to the magnetization (4.8) one has the gap of the out-of-plane spin waves, which cannot disappear due to the Kosterlitz–Thouless transition.

It also should be mentioned that the D-component vector model[9] considered in this article can be generalized for Hamiltonians with spin anisotropy. For example, one can consider the so-called n–D model,[10] in which only n from the total D spin components are coupled by the exchange interaction. In this sense the x–y model ($D=3$, $n=2$) is different from the plane rotator model ($D=n=2$). It should be noted that the quantities n and D play different roles: the well-known expansion of the critical indices of three-dimensional systems is an expansion in $1/n$, and the expansion developed here for low-dimensional systems is a $1/D$ expansion. The results of the present article can be generalized for the n–D model, as well as for more general models with anisotropic spin interactions.

APPENDIX. THE 1/*D* DIAGRAMS

The additional diagrams constituting $\hat{\Lambda}_{\alpha\alpha}^{(1/D)}(\mathbf{k})$ in the expression for the compact part of the transverse spin–spin correlation function $\tilde{\Lambda}_{\alpha\alpha}(\mathbf{k})$ of (2.4) represented in Fig. 2 have the following analytical form:

$$\hat{\Lambda}_{\alpha\alpha}^{(1)} = \tilde{\Lambda}_{\alpha\alpha zz} l_z \equiv \tilde{\Lambda}_{\alpha\alpha zz} \frac{1}{2} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \beta \tilde{J}_{z\mathbf{q}}$$

$$\hat{\Lambda}_{\alpha\alpha}^{(2)} = \frac{1}{2} \tilde{\Lambda}_{\alpha\alpha\beta\beta\gamma\gamma} \hat{\Lambda}_{\beta\beta\gamma\gamma} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} V_{\mathbf{q}} \tilde{V}_{\mathbf{q}}$$

$$\hat{\Lambda}_{\alpha\alpha}^{(3)} = \tilde{\Lambda}_{\alpha\alpha\beta\beta z} \hat{\Lambda}_{\beta\beta z} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \beta \tilde{J}_{z\mathbf{q}} \tilde{V}_{\mathbf{q}}$$

$$\hat{\Lambda}_{\alpha\alpha}^{(4)} = \frac{1}{2} \tilde{\Lambda}_{\alpha\alpha\beta\beta\gamma\gamma} \hat{\Lambda}_{\beta\beta z} \hat{\Lambda}_{\gamma\gamma z} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \beta \tilde{J}_{z\mathbf{q}} \tilde{V}_{\mathbf{q}}^2$$

$$\hat{\Lambda}^{(5)}_{\alpha\alpha}(\mathbf{k}) = \tilde{\Lambda}^2_{\alpha\alpha\beta\beta} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \beta\tilde{J}_{\mathbf{k}-\mathbf{q}} \tilde{V}_{\mathbf{q}}$$

$$\hat{\Lambda}^{(6)}_{\alpha\alpha}(\mathbf{k}) = \tilde{\Lambda}^2_{\alpha\alpha z} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \beta\tilde{J}_{\mathbf{k}-\mathbf{q}} \beta\tilde{J}_{z\mathbf{q}}$$

$$\hat{\Lambda}^{(7+7')}_{\alpha\alpha}(\mathbf{k}) = 2\tilde{\Lambda}_{\alpha\alpha\beta\beta} \hat{\Lambda}_{\alpha\alpha z} \hat{\Lambda}_{\beta\beta z} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \beta\tilde{J}_{\mathbf{k}-\mathbf{q}} \beta\tilde{J}_{z\mathbf{q}} \tilde{V}_{\mathbf{q}}$$

$$\hat{\Lambda}^{(8)}_{\alpha\alpha}(\mathbf{k}) = \tilde{\Lambda}_{\alpha\alpha\beta\beta} \hat{\Lambda}_{\alpha\alpha\gamma\gamma} \hat{\Lambda}_{\beta\beta z} \hat{\Lambda}_{\gamma\gamma z} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \beta\tilde{J}_{\mathbf{k}-\mathbf{q}} \beta\tilde{J}_{z\mathbf{q}} \tilde{V}^2_{\mathbf{q}} \quad \text{(A.1)}$$

Here $\tilde{\Lambda}_{\alpha\alpha\beta\beta} = \tilde{\Lambda}_{\beta\beta\gamma\gamma}$, $\tilde{\Lambda}_{\alpha\alpha z} = \tilde{\Lambda}_{\beta\beta z}$, etc., are the renormalized multispin cumulants with $\alpha \neq \beta \neq \gamma$ [no summation over β and γ in (A.1)] given by the formulas analogous to (2.5). As the diagrams (A.1) should be calculated only in the first nonvanishing order in $1/D$, one can use for the renormalized transverse interaction line $\beta\tilde{J}_{\mathbf{q}}$ (see Fig. 1c) the simplified expression $\beta\tilde{J}_{\mathbf{q}} \cong \beta J_{\mathbf{q}}/(1 - \tilde{\Lambda}_{\alpha\alpha}\beta J_{\mathbf{q}})$, where $\tilde{\Lambda}_{\alpha\alpha} = (\theta/D)G$ and G corresponds to the spherical model [see (2.21)]. The renormalized longitudinal interaction $\beta\tilde{J}_{z\mathbf{q}}$ (see Fig. 2b) is given by

$$\beta\tilde{J}_{z\mathbf{q}} = \frac{\beta J_{\mathbf{q}}}{1 - (\tilde{\Lambda}_{zz} + \tilde{\Lambda}^2_{\alpha\alpha z} \tilde{V}_{\mathbf{q}}) \beta J_{\mathbf{q}}} \quad \text{(A.2)}$$

and the renormalized four-spin correlation line $\tilde{V}_{\mathbf{q}}$ (see Fig. 2c) reads $\tilde{V}_{\mathbf{q}} = V_{\mathbf{q}}/(1 - \tilde{\Lambda}_{\alpha\alpha\beta\beta} V_{\mathbf{q}})$, where

$$V_{\mathbf{q}} = \frac{D}{2} v_0 \int \frac{d\mathbf{p}}{(2\pi)^d} \beta\tilde{J}_{\mathbf{p}} \beta\tilde{J}_{\mathbf{q}-\mathbf{p}} \quad \text{(A.3)}$$

is the unrenormalized four-spin correlation line, the factor D (or $D-1$, which plays no role here) in (A.3) resulting from the summation over the spin-component indices β and γ in the diagrams. Calculating now the renormalized cumulants $\tilde{\Lambda}$ in (A.1) in the lowest order in $1/D$ by the pass method, one gets[8]

$$\tilde{\Lambda}_{\alpha\alpha\beta\beta} \cong -\left(\frac{2}{D}\right)^3 \frac{1}{(2y)^2 (2y-1)}; \qquad \tilde{\Lambda}_{\alpha\alpha\beta\beta\gamma\gamma} \cong \left(\frac{2}{D}\right)^5 \frac{2(3y-1)}{(2y)^3 (2y-1)^3} \quad \text{(A.4)}$$

with $y \equiv 1/(\theta G)$ and, additionally,

$$\begin{aligned} \tilde{\Lambda}_{zz} &\cong \tilde{\Lambda}_{\alpha\alpha} + \tilde{\Lambda}_{\alpha\alpha\beta\beta}\xi^2; & \tilde{\Lambda}_{\alpha\alpha z} &\cong \tilde{\Lambda}_{\alpha\alpha\beta\beta}\xi \\ \tilde{\Lambda}_{\alpha\alpha zz} &\cong \tilde{\Lambda}_{\alpha\alpha\beta\beta} + \tilde{\Lambda}_{\alpha\alpha\beta\beta\gamma\gamma}\xi^2; & \tilde{\Lambda}_{\alpha\alpha\beta\beta z} &\cong \tilde{\Lambda}_{\alpha\alpha\beta\beta\gamma\gamma}\xi \end{aligned} \quad \text{(A.5)}$$

where $\xi = (D/\theta)(h + vm_0) = Dym_0$. With the use of these results for the corresponding contributions into $\Delta_{\mathbf{k}}^{(1/D)} \equiv \lim_{D\to\infty} [(D^2/\theta)\, \tilde{\Lambda}_{\alpha\alpha}^{(1/D)}(\mathbf{k})]$, one obtains

$$
\begin{aligned}
\Delta_{\alpha\alpha}^{(1)} &= -\frac{vG^2}{2y-1}\left[1 - 4m_0^2 y \frac{3y-1}{(2y-1)^2}\right] v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \tilde{\lambda}_{z\mathbf{q}} \\
\Delta_{\alpha\alpha}^{(2)} &= -2G^5 \frac{3y-1}{(2y-1)^4} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \varphi_{\mathbf{q}} \tilde{\varphi}_{\mathbf{q}} \\
\Delta_{\alpha\alpha}^{(3)} &= -8vm_0^2 yG^4 \frac{3y-1}{(2y-1)^4} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \tilde{\lambda}_{z\mathbf{q}} \tilde{\varphi}_{\mathbf{q}} \\
\Delta_{\alpha\alpha}^{(4)} &= 4vm_0^2 yG^6 \frac{3y-1}{(2y-1)^5} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \tilde{\lambda}_{z\mathbf{q}} \tilde{\varphi}_{\mathbf{q}}^2 \\
\Delta_{\alpha\alpha}^{(5)}(\mathbf{k}) &= \frac{2vG^4}{(2y-1)} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \tilde{\lambda}_{\mathbf{k}-\mathbf{q}} \tilde{\varphi}_{\mathbf{q}} \\
\Delta_{\alpha\alpha}^{(6+7+7'+8)}(\mathbf{k}) &= \frac{4m_0^2 yG^3}{(2y-1)^2} v_0 \int \frac{d\mathbf{q}}{(2\pi)^d} \tilde{\lambda}_{\mathbf{k}-\mathbf{q}} \tilde{\lambda}_{z\mathbf{q}} L_{\mathbf{q}}^2
\end{aligned}
\tag{A.6}
$$

where $\tilde{\varphi}_{\mathbf{q}} \equiv \varphi_{\mathbf{q}} L_{\mathbf{q}} \equiv \varphi_{\mathbf{q}}/(1 + a\varphi_{\mathbf{q}})$, $a \equiv G^2/(2y-1)$,

$$
\varphi_{\mathbf{q}} = v_0 \int \frac{d\mathbf{p}}{(2\pi)^d} \tilde{\lambda}_{\mathbf{p}} \tilde{\lambda}_{\mathbf{p}-\mathbf{q}}; \qquad \tilde{\lambda}_{\mathbf{q}} \equiv \frac{\tilde{J}_{\mathbf{q}}}{J_0} \equiv \frac{\lambda_{\mathbf{q}}}{1 - vG\lambda_{\mathbf{q}}} \tag{A.7}
$$

The expression for the renormalized z-interaction line $\tilde{\lambda}_{z\mathbf{q}} \equiv \tilde{J}_{z\mathbf{q}}/J_0$ can be written in the form

$$
\tilde{\lambda}_{z\mathbf{q}} = \frac{\lambda_{\mathbf{q}}}{1 - vG_{z\mathbf{q}}\lambda_{\mathbf{q}}}, \qquad G_{z\mathbf{q}} = G\left(1 - \frac{2m_0^2 y}{2y-1} L_{\mathbf{q}}\right) \tag{A.8}
$$

Further simplifications leading to the results listed at the end of the Section 2 can be archieved if one expresses $\varphi_{\mathbf{q}}$ and $\lambda_{\mathbf{q}}$ through $r_{\mathbf{q}}$ and $g_{\mathbf{q}}$ of (2.33) with the use of the relations

$$
\begin{aligned}
&(2y-1)(1 + a\varphi_{\mathbf{q}}) = \bar{r}_{\mathbf{q}} \equiv r_{\mathbf{q}} + 2m_0^2 y \\
&vG\tilde{\lambda}_{\mathbf{q}} = g_{\mathbf{q}} - 1; \qquad \tilde{r}_{\mathbf{q}} \equiv r_{\mathbf{q}} + 2m_0^2 yq_{\mathbf{q}} \\
&vG\tilde{\lambda}_{z\mathbf{q}} = g_{\mathbf{q}} - 1 - 2m_0^2 y(g_{\mathbf{q}} - 1)^2/\tilde{r}_{\mathbf{q}} \\
&vG\tilde{\lambda}_{z\mathbf{q}} L_{\mathbf{q}}^2 = (g_{\mathbf{q}} - 1)\frac{\bar{r}_{\mathbf{q}}}{\tilde{r}_{\mathbf{q}}} L_{\mathbf{q}}^2 = \frac{(2y-1)^2}{2m_0^2 y}\left[\frac{1}{\bar{r}_{\mathbf{q}}} - \frac{1}{\tilde{r}_{\mathbf{q}}}\right]
\end{aligned}
\tag{A.9}
$$

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